ALIAS–FREE PERIODIC SIGNAL ANALYSIS USING EFFICIENT RATE NONUNIFORM SAMPLING SETS

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ABSTRACT

In many applications such as signal integrity checking of hardware prototypes or determining dynamic behavior of wideband amplifiers etc. analysis of periodic radio signals with high accuracy is desired. The straight forward approach is to sample the signal to be analyzed with twice its highest harmonic content. But as fundamental frequencies are often far into the MHz range its harmonics may well span into the upper MHz or even GHz range. This leaves two possibilities: First, obey the sampling theorem and sample at that rate. But ADCs able to sample in the GHz range are expensive, power hungry and offer 6 to 8 bits maximum. Sometimes, this is not an option at all. Second, try to live with under sampling accepting alias frequencies in baseband. As long as aliased harmonics do not overlap this presents an acceptable solution. If they overlap, however, nothing can be done once the signal is sampled. But there is one more option presented in this paper, namely the possibility of deliberate nonuniform sampling. If the sampling set is chosen in a convenient way, mostly offered by an additive random sampling (ARS) scheme, this opens up promising possibilities to extend the alias–free processing range into the far MHz or even GHz region.

Index Terms— Nonuniform sampling, alias–free signal processing, least squares methods, radio signal processing, sampling methods

1. INTRODUCTION

Material presented in this paper is based on two concepts. First, the theory of sampling a signal nonuniformly a way that aliases are suppressed by the sampling process

$$\mathbf{s}\left(\omega^{(\mathbf{s})},t\right)=\sum_{n=0}^{N-1}\delta\left(t-\mathbf{t}_n\right). \tag{1}$$

Here $\mathbf{t}_n$ represents the *n*th sampling instance, a random variable, and $\omega^{(\mathbf{s})}$ designates a particular realization out of the set Ω_s of possible productions. In practice the number of samples taken N is bounded. The theory of sampling processes is extensively covered in [1] and [2] and it is shown in [2] that especially ARS is very effective

$$\mathbf{t}_n=\mathbf{t}_{n-1}+\mathbf{T}_I \tag{2}$$

in suppressing aliases. Therefore, an efficient hardware architecture has been designed realizing an ARS sampling driver (SD) for sampling radio signals nonuniformly. We use an inter–sample period stretching uniformly from T_{clk} to $2T_{clk}$ with f_{clk} being the clock rate of the SD. An efficient algorithm for sampling instant placement allowing for simple hardware realizations was introduced in [7] and is used to create the results presented here.

Second, the IEEE–STD–1057 four–parameter fitting algorithm [4]. Given a set of uniform samples this algorithm describes how the four parameters: amplitude, phase, DC offset and most importantly frequency of a sinusoid can be estimated based on an iterative least squares (LS) fit. Iterative, because the problem is nonlinear with respect to the frequency parameter estimation. An excellent description of the algorithm is found in [3].

When processing high frequency periodic signals (i. e. GHz range) harmonic content can often not be neglected due to signal impurities. Therefore the signals spectrum spreads much wider than the fundamental frequency. In such cases applying a four parameter fit delivers biased results. This was already discovered by Pintelon [9]. But Pintelon only considered uniform sampling. His analysis therefore suffers from potential alias overlap. We extend results obtained in [3] and [9] to nonuniform sampling sets. Also, in contrast to [3] not a pure sinusoid is being considered but rather any periodic, bandlimited signal with as much harmonics as fit into the digital alias–free signal processing (DASP) bandwidth B_{DASP} (cf. Fig. 1) determined by time quantum T_q, with $B_{DASP}=1/(2T_q)$.

2. SIGNAL MODELLING

Assume a particular realization of $\mathbf{s}(\omega^{(\mathbf{s})},t)$ did produce tuples $\{t_n,x_n\}$ of nonuniform sample times and associated signal values. The signal may be modeled by

$$x(t)=c_0+\sum_{n=1}^{N_{comp}}\left[c_{n,ev}\cos\left(n2\pi f_x\,t\right)+c_{n,od}\sin\left(n2\pi f_x\,t\right)\right] \tag{3}$$

were $N_{comp}-1$ is the number of modeled harmonics, f_x is the fundamental frequency and $c_{n,ev}$, $c_{n,od}$ represent the *n*th even or odd spectral coefficient, respectively. Clearly, in case of a pure sinusoid (3) simplifies into

$$x(t) = c_0 + c_{ev}\cos(2\pi f_x t) + c_{od}\sin(2\pi f_x t). \tag{4}$$

3. MODEL FITTING ALGORITHM

Starting with the sinusoid case given in (4) and assuming that the signals frequency is known it is shown in [3] that the parameter vector $\hat{\mathbf{c}} = (\hat{c}_0 \quad \hat{c}_{ev} \quad \hat{c}_{od})^{\mathrm{T}}$ is obtained by

$$\hat{\mathbf{c}} = \left(\mathbf{\Phi}\mathbf{\Phi}^{\mathrm{T}}\right)^{-1}\mathbf{\Phi}\mathbf{x} \tag{5}$$

with

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \cos(\omega_x t_0) & \cos(\omega_x t_1) & \cdots & \cos(\omega_x t_{N-1}) \\ \sin(\omega_x t_0) & \sin(\omega_x t_1) & \cdots & \sin(\omega_x t_{N-1}) \end{pmatrix} \tag{6}$$

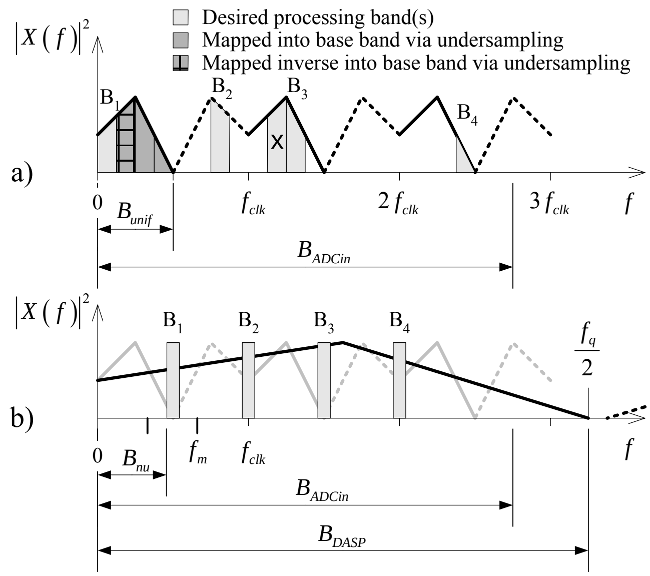

Fig. 1: Traditional undersampling a) versus DASP–sampling b) avoiding aliasing. Processing bands may be chosen such that arbitrary periodic signals can be analyzed.

(where $\omega_x = 2\pi f_x$) referred to as three–parameter fit in [4]. Note that the LS fit (5) works for uniform and nonuniform data sets. Surprisingly this was never stated explicitly but is important for the considerations in this paper. In practice f_x is not known exactly and to obtain most accurate results f_x must also be estimated! This leads to a four–parameter fit with vector $\hat{\mathbf{c}} = (\hat{c}_0 \quad \hat{f}_x \quad \hat{c}_{ev} \quad \hat{c}_{od})^{\mathrm{T}}$ creating a nonlinear problem. Since $x(t)$ is known at sampling times $\{t_n\}$ it can be approximated by a Taylor series using

$$\begin{aligned} \cos(\omega_x t_n) &\approx \cos(\hat{\omega}_x t_n) - t_n 2\pi\,\Delta f_x \sin(\hat{\omega}_x t_n) \\ \sin(\omega_x t_n) &\approx \sin(\hat{\omega}_x t_n) + t_n 2\pi\,\Delta f_x \cos(\hat{\omega}_x t_n) \end{aligned} \tag{7}$$

were $\Delta f_x = f_x - \hat{f}_x$ and $\hat{f}_x$ is an estimate close to the actual signal frequency. Inserting (7) into (4) delivers

$$\begin{aligned} x(t_n) &\approx c_0 + c_{ev}\cos(\hat{\omega}_x t_n) + c_{od}\sin(\hat{\omega}_x t_n) + \\ &\Delta f_x\left[-t_n 2\pi\, c_{ev}\sin(\hat{\omega}_x t_n) + t_n 2\pi\, c_{od}\cos(\hat{\omega}_x t_n)\right] \end{aligned} \tag{8}$$

which is still a nonlinear posed problem due to c_{ev} and c_{od} occurring in the corner bracket expression of (8). However, assuming further that close estimates $\hat{c}_{ev} \approx c_{ev}$ and $\hat{c}_{od} \approx c_{od}$ exist, (8) can be rewritten as

$$\begin{aligned} x(t_n) &\approx c_0 + c_{ev}\cos(\hat{\omega}_x t_n) + c_{od}\sin(\hat{\omega}_x t_n) + \\ &\Delta f_x\left[-t_n 2\pi\, \hat{c}_{ev}\sin(\hat{\omega}_x t_n) + t_n 2\pi\, \hat{c}_{od}\cos(\hat{\omega}_x t_n)\right] \end{aligned} \tag{9}$$

which renders problem (5) into a linear one. How close estimates can be determined is explained below. Note that by using (9) the search for vector $(\hat{c}_0 \quad \hat{f}_x \quad \hat{c}_{ev} \quad \hat{c}_{od})^{\mathrm{T}}$ has turned into a search for $(\hat{c}_0 \quad \Delta\hat{f}_x \quad \hat{c}_{ev} \quad \hat{c}_{od})^{\mathrm{T}}$ instead. This is not a problem because $f_x = \hat{f}_x + \Delta f_x$. But since we only have the estimate $\Delta\hat{f}_x$, parameter fitting turns into an iterative search with $\hat{f}_{x,i} = \hat{f}_{x,i-1} + \Delta f_{x,i-1}$ and (5) as well as (6) have to be applied repeatedly with updated parameter sets. The *i*th iteration parameter vector is referred to by

$$\hat{\mathbf{c}}_i = (\hat{c}_{0,i} \quad \Delta\hat{f}_{x,i} \quad \hat{c}_{ev,i} \quad \hat{c}_{od,i})^{\mathrm{T}} \tag{10}$$

With iteration index *i* estimation matrix $\mathbf{\Phi}$ becomes

$$\mathbf{\Phi}_i = \begin{pmatrix} 1 & \cdots & 1 \\ 2\pi t_0\left[-\hat{c}_{ev,i}\sin(\hat{\omega}_{x,i} t_0) + \hat{c}_{od,i}\cos(\hat{\omega}_{x,i} t_0)\right] & \cdots & 2\pi t_{N-1}\left[-\hat{c}_{ev,i}\sin(\hat{\omega}_{x,i} t_{N-1}) + \hat{c}_{od,i}\cos(\hat{\omega}_{x,i} t_{N-1})\right] \\ \cos(\hat{\omega}_{x,i} t_0) & \cdots & \cos(\omega_{x,i} t_{N-1}) \\ \sin(\hat{\omega}_{x,i} t_0) & \cdots & \sin(\omega_{x,i} t_{N-1}) \end{pmatrix} \tag{11}$$

The considerations made so far are applicable in case of a periodic signal (3) too. Harmonics being integer multiples of f_x approximation only needs to take into account the adaptation to the fundamental frequency Δf_x. Thus the *i*th iteration vector turns into

$$\hat{\mathbf{c}}_i = \left(\hat{c}_{0,i} \quad \Delta\hat{f}_{x,i} \quad \hat{c}_{1,ev,i} \quad \hat{c}_{1,od,i} \quad \cdots \quad \hat{c}_{N_{comp},ev,i} \quad \hat{c}_{N_{comp},od,i}\right)^{\mathrm{T}}. \tag{12}$$

Similarly the *i*th estimation matrix $\mathbf{\Phi}_i$ is obtained by

$$\mathbf{\Phi}_i = \begin{pmatrix} 1 & \cdots & 1 \\ 2\pi t_0\left[-\hat{c}_{ev,i}\sin(\hat{\omega}_{x,i} t_0) + \hat{c}_{od,i}\cos(\hat{\omega}_{x,i} t_0)\right] & \cdots & 2\pi t_{N-1}\left[-\hat{c}_{ev,i}\sin(\hat{\omega}_{x,i} t_{N-1}) + \hat{c}_{od,i}\cos(\hat{\omega}_{x,i} t_{N-1})\right] \\ \cos(\hat{\omega}_{x,i} t_0) & \cdots & \cos(\omega_{x,i} t_{N-1}) \\ \sin(\hat{\omega}_{x,i} t_0) & \cdots & \sin(\omega_{x,i} t_{N-1}) \\ \vdots & \ddots & \vdots \\ \cos(N_{comp}\hat{\omega}_{x,i} t_0) & \cdots & \cos(N_{comp}\omega_{x,i} t_{N-1}) \\ \sin(N_{comp}\hat{\omega}_{x,i} t_0) & \cdots & \sin(N_{comp}\omega_{x,i} t_{N-1}) \end{pmatrix} \tag{13}$$

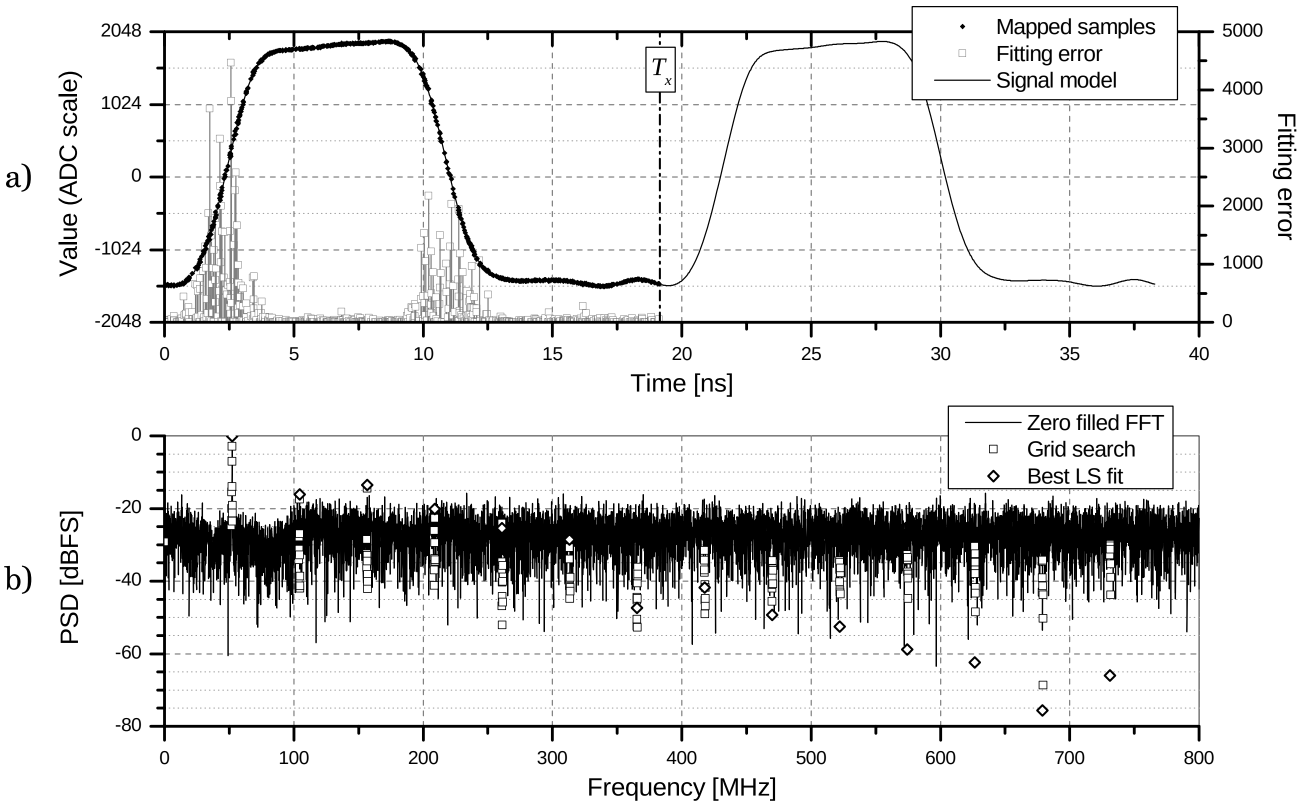

Fig. 2: Fitting of a rect–like signal generated by a PHILIPS PM5786B pulse generator. Estimated fundamental period T_x is 19.1 ns. Measurement interval is 10.24 µs, thus, 534 signal periods were used for curve fitting. Number of nonuniform samples is $N_{nu} = 679$. Shown are a) the nonuniform samples mapped into the fundamental period and b) the zero filled FFT spectrum as well as the grid search results along with the parameters obtained with the last iteration step (power values).

Fortunately the approximation procedure to arrive at the final result remains the same thus the approximation algorithm can be summarized as given in Tab. 1.

init	• Zero filled FFT (cf. [8]) to get fundamental component parameters $\hat{c}_{ev}$, $\hat{c}_{od}$ and $\hat{f}_x$ • Use $\hat{f}_x$ constructing initial frequency grid search matrix $\mathbf{\Phi}_{init}$ (see Section 4). Use (5) to find initial estimate parameters $\hat{c}_{1,ev,0}$, $\hat{c}_{1,od,0}$ $\cdots$ $\hat{c}_{N_{comp},ev,0}$, $\hat{c}_{N_{comp},ev,0}$. $i \leftarrow 0$, $\Delta\hat{f}_x \leftarrow \infty$, $\hat{\mathbf{c}}_0 \leftarrow (0 \quad 0 \quad \hat{c}_{1,ev,0} \quad \hat{c}_{1,od,0} \quad \cdots \quad \hat{c}_{N_{comp},ev,0} \quad \hat{c}_{N_{comp},od,0})^{\mathrm{T}}$ Use (13) to construct $\mathbf{\Phi}_0$
1)	**WHILE** $\left\|\Delta\hat{f}_x\right\| > \varepsilon$ **DO**
2)	$\hat{\mathbf{c}}_{i+1} \leftarrow (\mathbf{\Phi}_i \mathbf{\Phi}_i^{\mathrm{T}})^{-1} \mathbf{\Phi}_i \mathbf{x}$, $\quad \Delta\hat{f}_{x,i+1} \leftarrow \hat{\mathbf{c}}_{i+1}[1]$
3)	$\hat{f}_{x,i+1} \leftarrow \hat{f}_{x,i} + \Delta\hat{f}_{x,i+1}$
4)	Use (13) to build $\mathbf{\Phi}_{i+1}$ with $\hat{\mathbf{c}}_{i+1}$ and $\hat{f}_{x,i+1}$
5)	$i \leftarrow i+1$
	NEXT
	Result is in $\hat{\mathbf{c}}_i$

Tab. 1: Periodic signal fitting using nonuniform sampling sets. Note: $\leftarrow$ indicates assignment.

The heuristic threshold ε is subject to change with the analysis setup as it depends on factors like SNR and number of samples used in calculations. Therefore, no fixed value can be given and ε has to be chosen according to the measurement task at hand.

4. FINDING INITIAL ESTIMATES

Finding initial estimates of (12) close to the true coefficients is vital for fast convergence. It is more involved for the nonuniform than in the uniform case. This is the prize to be paid for alias–free processing.

We use a two step method: First, a zero stuffed (as opposed to zero padded) FFT is applied to obtain first phase, amplitude and frequency estimates of the fundamental frequency (cf. [8]). The harmonics usually drown in noise at this point since the spectral dynamic range is about 20 dB (comprehensive simulations are found in [10]). To obtain a complete set of estimates we perform a frequency grid search. A matrix $\mathbf{\Phi}_{init}$ similar to (13) but without Δf_x terms is constructed for frequencies near the expected fundamental as well as the harmonics using FFT grid frequencies. This is shown in Fig. 2b. We use 10 grid points around every bin we suspect to be part of the signal. This is done throughout the alias–free range $[0,1/(2T_q)]$. Construction of such matrixes is well presented in [5]. Our hardware realizes a time quantum of 625 ps. So we search in the range DC … 800 MHz. With matrix $\mathbf{\Phi}_{init}$ we perform a LS fit (5) to arrive at estimates to seed initial vector $\hat{\mathbf{c}}_0$ with. This vector is used in the algorithm described in Tab. 1 for further parameter refinement.

5. PRACTICAL RESULTS

To verify our proposed algorithm in an actual radio signal processing system we collected nonuniform samples with our prototype hardware. The prototype features two sampling channels with an AD9433 (12 Bit). We use channel–A only here. The mean sampling rate f_m is 66 MHz. The analogue input bandwidth B_{ADCin} of the converters is 750 MHz so they are well suited for undersampling or application of DASP, respectively. The analyzed signal was delivered by a PHILIPS PM5786B pulse generator. A representative excerpt of our results is depicted in Fig. 2.

The mapping of all nonuniform samples into fundamental period T_x becomes possible only after accurate determination of T_x! The error between mapped samples and the signal model (3) is given by

$$e(t_n') = \left(x(t_n') - x_n\right)^2 \qquad (14)$$

with $t_n' = t_n \bmod T_x$. It is depicted in Fig. 2a. As expected it is largest at steeper parts of the signal reflecting the fact that sampling time placement errors surface mainly at steep parts of the signal.

We analyzed up to the 13$^{\text{th}}$ harmonic. Given the estimated fundamental of 52.2 MHz this means a total bandwidth of 730.8 MHz. Traditionally, a sampling frequency of 1.5 GHz is needed to obtain this processing bandwidth. This is a maximum requirement. It is inefficient because much more information is collected than required. Why? Because the signal is present only in very isolated parts of the spectrum. It therefore contains much less information. Our method utilizes this fact and brings the (average) sampling rate down to 66 MHz. Thus, much less information is gathered per time interval, yet, still enough to reconstruct the original signal (indicated in Fig. 1b). This is why we refer to our method as efficient rate nonuniform sampling.

With our measurements we obtain convergence after five iterations. A threshold $\varepsilon = 30$ Hz suffices in our case.

6. CONCLUSIONS

The application of deliberate nonuniform sampling to periodic signals leads to efficient sampling sets whose density is matched far better to the signals' information rate than a brute force approach using a sampling rate $f_s = 2 f_{h,max}$, $f_{h,max}$ being the maximum harmonic considered for signal analysis. A different approach for radio signals was suggested in [9] using undersampling accepting aliases. If aliases overlap, this method fails. Our solution avoids this problem by using sets of nonuniform samples designed especially for alias suppression. ARS has proven very effective in this regard.

Although we have tested this method up to frequencies of 800 MHz it is applicable to much wider bands. A time quantum as small as 10 ps (cf. [6]) would open up a direct digital alias–free processing range as far as 50 GHz with this methodology.

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