OPTIMAL SAMPLING FUNCTIONS IN NONUNIFORM SAMPLING DRIVER DESIGNS TO OVERCOME THE NYQUIST LIMIT

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ABSTRACT

In some applications the observed samples are inherently nonuniform. In contrast to that in this paper we take advantage of deliberate nonuniform sampling and perform DSP where the classical approaches leave off. For instance think about mobile communication or digital radio. Deliberate nonuniform sampling promises increased equivalent sampling rates with reduced overall hardware costs. The equivalent sampling rate is the sampling rate that a uniform sampling device would require in order to achieve the same processing bandwidth. While the equivalent bandwidth of a realizable system may well extend into the GHz range its mean sampling rate is usually in the MHz range. Current existing prototype systems achieve 40 times the bandwidth of a classic DSP system that would operate uniformly (cf. [3] and [4]). Throughout the literature on nonuniform sampling (e. g. [1], [2] and [5]) many sampling schemes have been investigated. In this paper the authors discuss a nonuniform sampling scheme that is especially suited to be implemented in digital devices, thus, fully exploiting state-of-the-art ADCs without violating their specifications. An analysis of the statistical properties of the algorithm is given to demonstrate common pitfalls and to prove its correctness.

1. INTRODUCTION

At the heart of a deliberate nonuniform sampling device there is the sampling driver (SD) core generating the sampling pulse train used to digitize the analog signal. To realize a SD in digital circuits obviously a synchronous design is desirable. According to sampling theory [1] a straightforward implementation of a SD core produces periodic sampling with jitter. A pseudo random number generator (PRNG) usually generates numbers that are passed to a counter that when digitally controllable delay line (DCDL) together with sampling pulses produced by a central control unit. Every digital circuit, driving an ADC, performs periodic sampling with jitter due to phase noise that is always present. However, a simple SD core realization depicted in Fig. 3 (without grayed elements) does it deliberately. The time axis can be thought of as being separated into time slots having system clock duration T_{clk}. Inside every slot a sampling instance t_k is produced. It is important to note that a vital property of a successful SD design must realize equal probability to produce the sampling point anywhere in the *k*-th time slot. Then the density of sampling points is equal anywhere along the time axis. Failure to do so will result in an undesired spectrum of the sampled signal containing spurious frequencies, a result of the convolution of the spectra of the sampling process and the analog signal.

A real design will not be able to produce sampling points at arbitrary moments in time but will rather realize time increments of so called time quantum size T_Q. The equivalent sampling rate then is given by

$$f_Q = \frac{1}{T_Q}\,. \tag{1}$$

Because of the time quantum the PDF of a particular sampling instance becomes a discrete PDF as depicted in Fig. 1. Since there is a limited amount of time increments within one time slot one can define the system clock period to time quantum ratio M

$$M = \frac{T_{clk}}{T_Q}\,. \tag{2}$$

M is a key parameter of a sampling driver since it represents the factor about which the processing bandwidth of the digital system is increased if sampling takes place like described in equation (3).

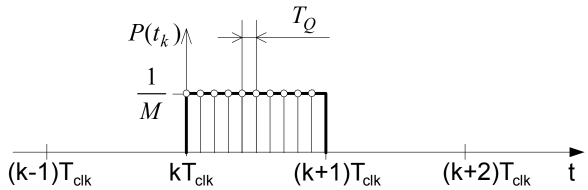

Fig. 1: PDF of sampling instances at k-th time slot.

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It is convenient to keep M a power of two. The process of sampling instant generation is well known as periodic sampling with jitter (cf. [1]) and can be described by

$$t_k = kT_{clk} + \varepsilon_k T_Q \qquad k, \varepsilon_k \in \mathbf{N} \qquad 0 \le \varepsilon_k < M \tag{3}$$

where ε_k is a pseudo random number produced by the PRNG at the k-th time slot. Unfortunately, equation (3) has an undesirable property. Two successive samples may be separated by only the time quantum T_Q. To further quantify this we make the following derivations. Let T_s be the time between to successive samples, the intersample time

$$T_s = t_k - t_{k-1} . \tag{4}$$

Thus the intersample time is a derived random variable. For convenience we define the Laplacian random experiment E_0

$$\begin{aligned} &\Omega_0 = \{\omega_0^{(0)}, \omega_1^{(0)}, ..., \omega_n^{(0)}, ..., \omega_{M^2-1}^{(0)}\} \\ &\Omega_0 = \{(0,0),(0,1),...(i_n, j_n),...,(M-1,M-2),(M-1,M-1)\} \\ &i_n, j_n, n \in \mathbf{N}; \quad 0 \le i_n, j_n, n < M \\ &\omega_n^{(0)} \equiv (i_n, j_n) \equiv \quad (\varepsilon_{k-1} = i_n \quad and \quad \varepsilon_k = j_n) \\ &P(\omega_n^{(0)}) = \frac{1}{M^2} \end{aligned} \tag{5}$$

Here (i_n, j_n) denotes the event that ε_{k-1} takes on value i_n and ε_k takes on value j_n. It is easy to see that there are M^2 such events. Assuming that both ε_{k-1} and ε_k have uniform distribution and are statistically independent, it immediately follows that the events (i_n, j_n) have equal probability $1/M^2$. One can now define a different random experiment E_1 with a set Ω_1 of $2M$ elementary events

$$\begin{aligned} &\Omega_1 = \{\omega_0^{(1)}, \omega_1^{(1)}, ..., \omega_l^{(1)}, ..., \omega_{2M-1}^{(1)}\} \\ &\Omega_1 = \{0, T_Q, 2T_Q, ..., lT_Q, ..., (2M-1)T_Q\} \qquad l \in \mathbf{N}, 0 \le l < 2M \\ &\omega_l^{(1)} \equiv T_s = lT_Q \end{aligned} \tag{6}$$

where the l-th event in Ω_1 denotes the event that T_s takes on value lT_Q. Unlike the events in Ω_0 the events in Ω_1 do not occur with equal probability. However, these probabilities can be obtained from E_0 by

$$\begin{aligned} &P(\omega_l^{(1)}) = \sum_{\omega_n^{(0)} \in \omega_l^{(1)}} P(\omega_n^{(0)}) = \frac{1}{M^2} \sum_{\omega_n^{(0)} \in \omega_l^{(1)}} 1 \\ &n, l \in \mathbf{N}; \quad 0 \le n < M^2; \quad 0 \le l < 2M \end{aligned} \tag{7}$$

But when is an event in Ω_0 said to be a favorable event in terms of an event in Ω_1? Fortunately using (2), (3) and (4) this is easily stated as

$$\omega_n^{(0)} \in \omega_l^{(1)} \qquad if \qquad l = j_n + M - i_n . \tag{8}$$

Applying (7) and (8) one can calculate probabilities for all events in Ω_1 and hence the discrete PDF of T_s. It assumes the discrete PDF sketched in Fig. 2.

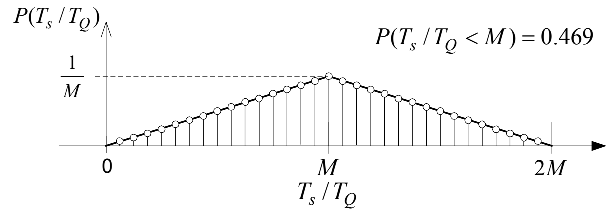

Fig. 2: PDF of intersample time.

The system clock period T_{clk} is usually matched to the minimum conversion time of the attached ADC

$$T_{clk} = \min\{T_{ENCODE}\} . \tag{9}$$

This is justified by the design decision to operate the sampling driver also in a uniform mode (ε_k constant) in which case the minimum ADC sampling period should be fully utilized. The intersample time constraint

$$\begin{aligned} t_k - t_{k-1} &\ge T_{clk} \\ T_s &\ge MT_Q \end{aligned} \tag{10}$$

must always be met. Given (7) and (8) we can calculate the probability that (10) is not met. In case of this straightforward design the probability to violate the constraints is 47%. Since this probability is non-zero we conclude that such a design is unusable as a SD.

2. PHASE SHIFTING

In this Section we propose a sampling scheme that deliberately introduces phase shifts at times when consecutive samples occur too close for the ADC to handle.

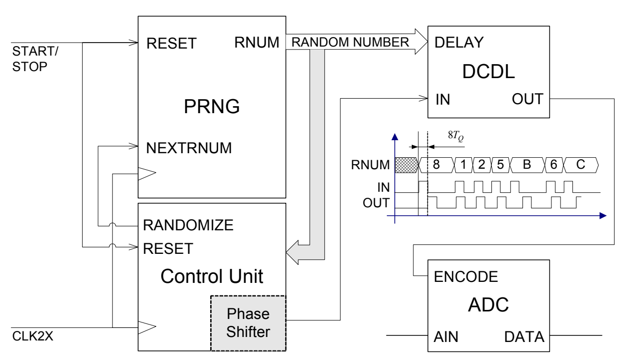

Fig. 3: Deliberate (phase shift) SD building block.

Fig. 3 shows the modified SD building block. A phase shift of the sampling pulse means that it is deferred one SD system clock period (i. e. 360°). The modified sampling scheme can be described recursively as

$$t_k = t_{k-1} - \varepsilon_{k-1} T_Q + T_{clk} + \begin{Bmatrix} 0 & if \quad \varepsilon_{k-1} < \dfrac{M}{2} \\ T_{clk} & otherwise \end{Bmatrix} + \varepsilon_k T_Q . \qquad (11)$$

$$k, \varepsilon_k \in \mathbf{N}; \quad 0 \le \varepsilon_k < M$$

The process of phase shifting fundamentally changes the character of the sampling scheme. What was periodic sampling with deliberate jitter before becomes additive random sampling. This is also expressed by the fact that the index of the sampling instance *k* is no longer identical to the index of the system clock *m* (see Fig. 4).

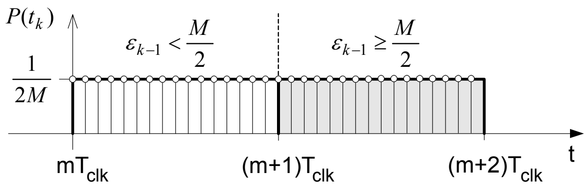

Fig. 4: PDF of sampling instance with phase shift.

Fortunately, any additive random sampling scheme will produce a sampling point density function (SPDF) that approaches a constant value after a transient phase. This property (based on the central limit theorem) is stated here without further proof and the interested reader is referred to [1] for a detailed treatment. We verify this property for the proposed algorithm in Section 4. Due to the randomly occurring phase shift the PDF of a particular sampling instance is stretched like shown in Fig. 4. The PDF of the derived random variable T_s looks different than that obtained in Section 1. First, using (4) and (11) one can write

$$T_s = \left(M - \varepsilon_{k-1} + \begin{Bmatrix} 0 & if \quad \varepsilon_{k-1} < \dfrac{M}{2} \\ M & otherwise \end{Bmatrix} + \varepsilon_k \right) T_Q . \qquad (12)$$

$$k, \varepsilon_k \in \mathbf{N}; \quad 0 \le \varepsilon_k < M$$

Observing that (12) will never deliver a period greater than $3M$, for convenience, an experiment E_2 with a set Ω_2 of $3M$ elementary events is defined

$$\begin{aligned} &\Omega_2 = \{\omega_0^{(2)}, \omega_1^{(2)}, ..., \omega_l^{(2)}, ..., \omega_{3M-1}^{(2)}\} \\ &\Omega_3 = \{0, T_Q, 2T_Q, ..., lT_Q, ..., (3M-1)T_Q\} \\ &\omega_l^{(2)} \equiv T_s = lT_Q \\ &l \in \mathbf{N}; \quad 0 \le l < 3M \end{aligned} . \qquad (13)$$

Since the underlying random process is the process that generates events in Ω_0 one can determine the probabilities of events in Ω_2 a priori

$$\begin{aligned} &P(\omega_l^{(2)}) = \sum_{\omega_n^{(0)} \in \omega_l^{(2)}} P(\omega_n^{(0)}) \qquad n, l \in \mathbf{N}, 0 \le n < M^2, 0 \le l < 3M \\ &\omega_n^{(0)} \in \omega_l^{(2)} \quad if \quad l = M - i_n + \begin{Bmatrix} 0 & if \quad i_n < \dfrac{M}{2} \\ M & otherwise \end{Bmatrix} + j_n \end{aligned} . \qquad (14)$$

The probabilities obtained are depicted in Fig. 5. As can be seen from these calculations the probability to produce intersample times less than MT_Q is around 11% and thus non-zero.

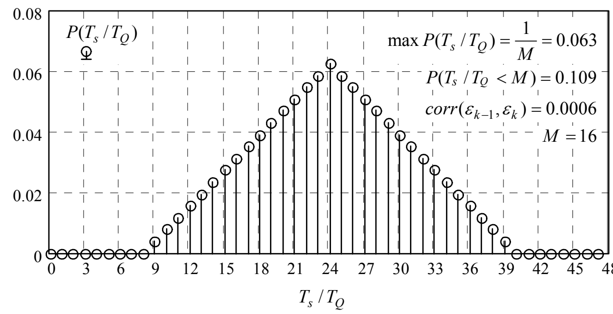

Fig. 5: PDF of intersample time with phase shift.

The property of such a sampling driver is certainly better but would still be too demanding for an attached ADC being operated at its limits as described above.

In the next Section we pursue a way to obtain a sampling driver algorithm that will deliver intersample times never violating the constraints set by (9) and (10).

3. RANDOM NUMBER CORRELATION

Until now it was assumed that two consecutive numbers produced by the PRNG are statistically independent. While this seems desirable at first glance we will show here that correlation between random numbers can be beneficial.

When generating pseudo random numbers maximum length linear feedback shift registers (LFSR) are very commonly used (see [6]). These registers seem to be very suitable to be incorporated into the implementation of a sampling driver. Using a slice of bits from a long LFSR one can write for two consecutive random numbers ε_{k-1} and ε_k

$$\varepsilon_k = (2\varepsilon_{k-1} + \tau_k) \bmod 2^n \qquad k, \varepsilon_k \in \mathbf{N} \qquad \tau_k \in \{0,1\} \qquad (15)$$

where τ_k is a binary random number assumed to be evenly distributed and *n* is the dimension of the vector passed as random number to the DCDL. It is important to note that, given (15), the probabilities for events in Ω_0 are no longer evenly distributed. Some events $\omega^{(0)}$ may even never be observed. However, it is easy to determine the distribution of T_s through simulations when two successive random numbers are correlated as given by (15). The resulting PDF is shown in Fig. 6.

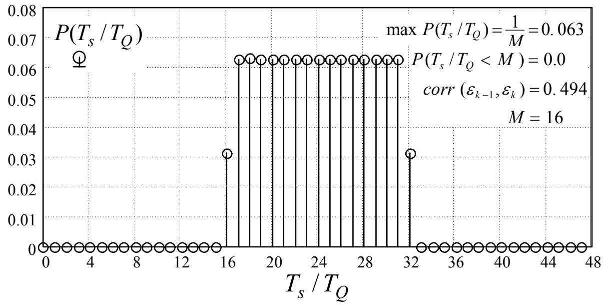

Fig. 6: PDF of intersample time with phase shift and correlated random numbers.

The simulation results clearly reveal that this time the PDF of the intersample time satisfies the constraints set by (9) and (10).

4. ASSESSING THE SPDF

While the distribution of the intersample time is a very important property it is necessary to assess the generated sampling point density function as well

$$spdf(t) = \sum_{k=0}^{\infty} P(t_k(t)). \quad (16)$$

As said in the introduction, when designing the sampling algorithm of a SD, the SPDF, which determines how samples are placed on the time axis, should be a const function. This is due to the convolution in the frequency domain of both, the spectrum of the SPDF and the spectrum of the sampled signal. To check for the properties of the generated SPDF the sampling pulse train for every possible seed of the PRNG was recorded and thus we obtained an estimate of the SPDF produced by the proposed algorithm. The results are presented in Fig. 7.

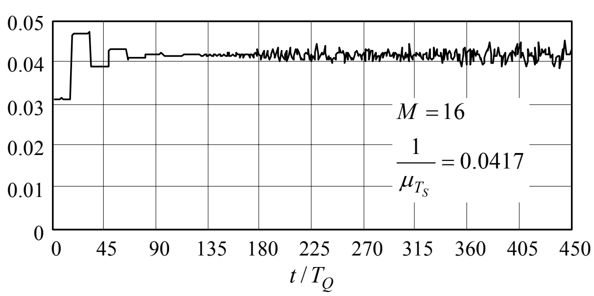

Fig. 7: Estimated discrete SPDF.

After a transient phase the SPDF remains constant at a value given by the inverse of the mean intersample time. This result confirms theoretical considerations given e. g. in [1]. The noise observed in Fig. 7 is easily explained because of the deliberate correlation of the random numbers taken to produce the sampling pulse trains. Because of this not all combinations of random numbers can be observed and a deviation from the perfect const line is the result.

CONCLUSIONS

In this paper we have derived an efficient sampling algorithm for deliberate random sampling. The type of nonuniform sampling performed must be classified as additive random sampling. The sampling point density generated by the algorithm has a const value. It is given by the inverse of the mean intersample time. The algorithm is well suited to fully utilize the conversion rate of an ADC (i. e. if the sampling drivers system clock is matched to the minimum conversion time given by the specification of the ADC). The algorithm is easily implemented in hardware.

It was shown that introducing deliberate correlation into the process of random number generation has a beneficial effect. It will create exactly the sampling pulse train that best utilizes a given ADC. In our case the introduced correlation coefficient between two consecutive random numbers is 0.5.

The DCDL solution presented here is superior to other architectures that use only synchronous logic to generate sampling pulses. The processing bandwidth of such systems is bound to half of the system clock rate demanding for much higher clock rates. In contrast to that the bandwidth of a system presented here is given by $1/2T_Q$. Time quantum steps in the range of Pico seconds can be realized.

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